Gaussian Distribution (Multivariate)

$x \in \mathbb{R}^D, \quad \mu \in \mathbb{R}^D, \quad \Sigma \in \mathbb{R}^{D \times D}$

$$p(x) = \frac{1}{\sqrt{2\pi}^D \, |\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right)$$

$$\sim \boxed{N(\mu, \Sigma)}$$

① Σ is symmetric, also p.d.

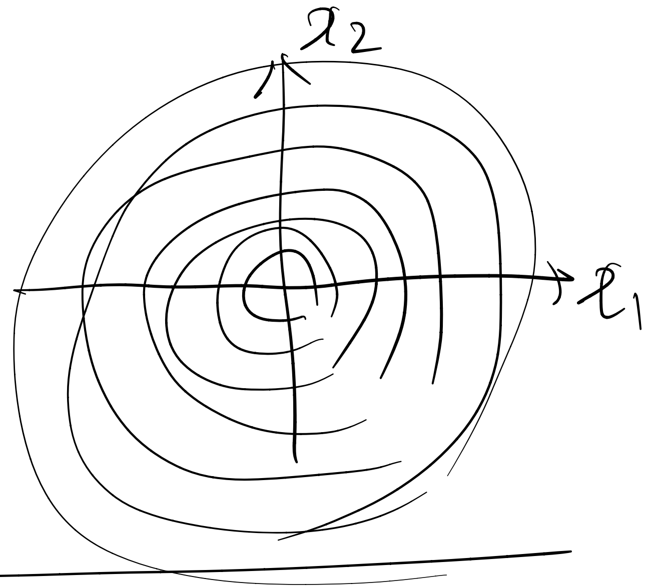
$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & \cdots & \sigma_{1D} \\ & \ddots & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \sigma_D^2 \end{bmatrix}$$

$\Sigma = \Sigma^T$

$\sigma_{ij} = \sigma_{ji}$

④ Isotropic Gaussian

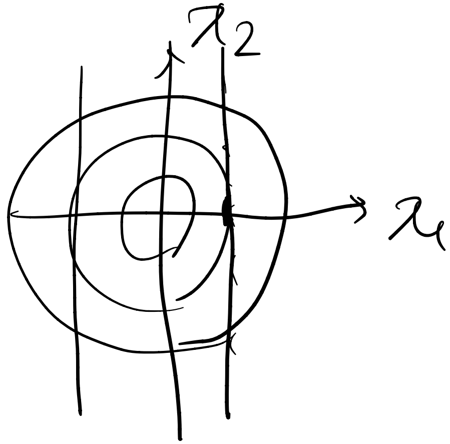
$$\Sigma = I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & 1 \end{bmatrix}$$



$\Sigma \in \mathbb{R}^{2\times 2}$

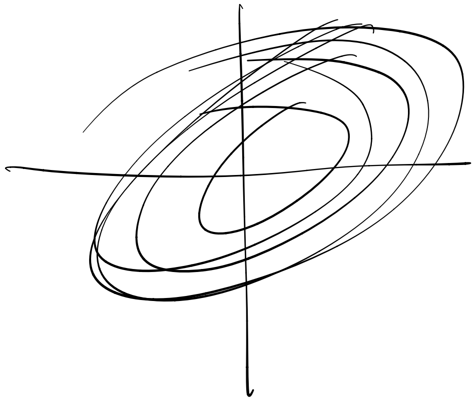
$$\Sigma = \begin{pmatrix} 1 & \sigma_{12} \\ \sigma_{21} & 1 \end{pmatrix}$$

① $\sigma_{12} = 0$ $\quad x_1 \perp\!\!\!\perp x_2$

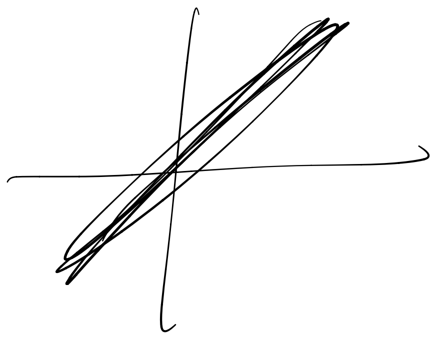


$p(x_2 \mid x_1) = p(x_2)$

② $0 < \sigma_{12} < 1$



② $\sigma_{12} \simeq 1$



$$p(x) = p(x_1, x_2) = \frac{1}{\sqrt{2\pi}^{D} |\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right).$$

$$\boxed{\Sigma = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}} \quad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \qquad \boxed{\begin{pmatrix} \frac{1}{\lambda_1} & 0 \\ 0 & \frac{1}{\lambda_2} \end{pmatrix}}$$

$$= \frac{1}{\sqrt{2\pi}^{D} |\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(x_1-\mu_1)\frac{1}{\lambda_1}(x_1-\mu_1) - \frac{1}{2}(x_2-\mu_2)\frac{1}{\lambda_2}(x_2-\mu_2)\right)$$

$$= C \exp\left(-\frac{1}{2}\frac{(x_1-\mu_1)^2}{\lambda_1}\right) \exp\left(-\frac{1}{2}\frac{(x_2-\mu_2)^2}{\lambda_2}\right)$$

$$= p(x_1)\, p(x_2) \quad \Rightarrow \quad \boxed{x_1 \perp\!\!\!\perp x_2}$$

$$p(x) = \frac{1}{\sqrt{2\pi}^{D} \, |\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(x-\mu)^{T}\Sigma^{-1}(x-\mu)\right)$$

$$\sim \boxed{N(\mu, \Sigma)}$$

$$a_1 x_1^2 + a_2 x_2^2 + a_3 x_1 x_2$$
$$+ a_4 x_1 x_3 + \cdots$$

$\rightarrow$ quadratic function

w.r.t X.

$$p(x_1, x_2) = c \exp\left(-\frac{1}{2}(a_{11} x_1^2 + a_{12} x_1 x_2 + a_{22} x_2^2)\right)$$

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}.$$

(perfect square form)

$p(x_1 \mid x_2)$

$$p(x_1, x_2) = C \exp\left(-\frac{1}{2}\left\{a_{11}\left(x_1 + \frac{a_{12}x_2}{2a_{11}}\right)^2 + a_{22}x_2^2 - \frac{a_{12}^2 x_2^2}{4a_{11}^2}\right\}\right)$$

$$= C \cdot \exp\left(-\frac{1}{2}a_{11}\left(x_1 + \frac{a_{12}x_2}{2a_{11}}\right)^2\right) \exp\left(\overset{-\frac{1}{2}}{\downarrow} a_{22}x_2^2 - \frac{a_{12}^2 x_2^2}{4a_{11}^2}\right)$$

$$\underline{p(x_1 | x_2) = C' \exp\left(-\frac{1}{2}a_{11}\left(x_1 + \frac{a_{12}}{2a_{11}}x_2\right)^2\right)}$$

From original multivariate Gaussian Form.

$$p(x) = N(\mu, \Sigma) = \frac{1}{\sqrt{2\pi}^{D} |\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right)$$

$$X \in \mathbb{R}^D \qquad X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \qquad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \qquad \Sigma = \left[\begin{array}{c|c} \Sigma_1 & \Sigma_{12} \\ \hline \Sigma_{21} & \Sigma_2 \end{array}\right]$$

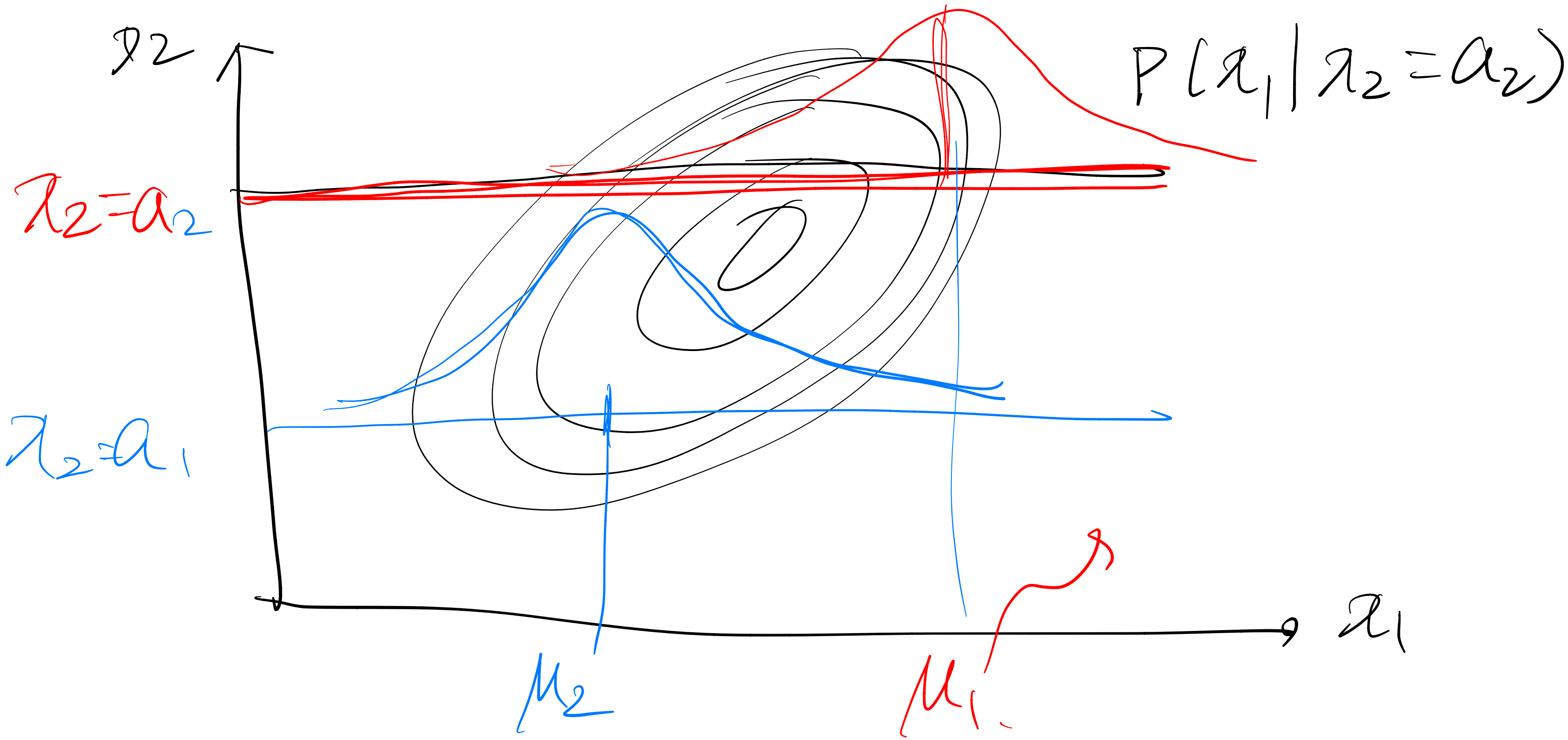
$$= \frac{1}{\sqrt{2\pi}^{D} \left| \begin{pmatrix} \Sigma_1 & \Sigma_{12} \\ \Sigma_{21} & \Sigma_2 \end{pmatrix} \right|^{\frac{1}{2}}} \exp\left(-\frac{1}{2} [x_1^T - \mu_1^T, x_2^T - \mu_2^T] \begin{bmatrix} \Sigma_1 & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}^{-1} \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{pmatrix}\right)$$

$$\Rightarrow P(X_1 | X_2) = N(\mu_{1|2}, \Sigma_{1|2})$$

$$p(x_1 \mid x_2) = N(\mu_{1|2}, \Sigma_{1|2})$$

$$\mu_{1|2} = \mu_1 + \Sigma_{12} \Sigma_2^{-1} (x_2 - \mu_2)$$

$$\Sigma_{1|2} = \Sigma_1 - \Sigma_{12} \Sigma_2^{-1} \Sigma_{21}.$$



Gaussian Marginal p.d.f.

$$p(X) = p(x_1, x_2) \Rightarrow p(x_1) = \int p(x_1, x_2)\, dx_2$$

$$p(x_1, x_2) = C \exp\left(-\frac{1}{2}\left(a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2\right)\right)$$

$$p(x_1) = \int p(x_1, x_2)\, dx_2 = N(\mu_1, \Sigma_1)$$

$$X = \begin{pmatrix} X_1 \\ \hline X_2 \end{pmatrix} \quad \mu = \begin{pmatrix} \mu_1 \\ \hline \mu_2 \end{pmatrix} \quad \Sigma = \begin{bmatrix} \Sigma_1 & \Sigma_{12} \\ \Sigma_{21} & \Sigma_2 \end{bmatrix}$$

$$p(x_1) = \int p(x_1, x_2)\,dx_2$$

$$= \int c \exp\left(-\frac{1}{2} a_{22}\left(x_2 - \frac{a_{12}x_1}{2a_{22}}\right)^2 + \underbrace{a_{11}x_1^2 - \frac{a_{12}^2 x_1^2}{4a_{22}^2}}_{\text{not relate to } dx_2.}\right) dx_2$$

$$= c \exp\left(-\frac{1}{2}\left(a_{11}x_1^2 - \frac{a_{12}^2 x_1^2}{4a_{22}^2}\right)\right) \underbrace{\int \exp\left(-\frac{1}{2} \text{——}\right) dx_2}_{\text{Constant}}$$

$$= c' \exp\left(-\frac{1}{2}\left(a_{11} - \frac{a_{12}^2}{4a_{22}^2}\right) x_1^2\right)$$

$\therefore$ Gaussian. $\boxtimes$