"Learning"

$x_1, x_2, \ldots, x_N \in \mathbb{R}^D$

$y_1, y_2, \ldots, y_N \in \mathbb{R}$

$$z = \begin{pmatrix} | \\ x \\ | \\ y \end{pmatrix} \in \mathbb{R}^{D+1}$$

① model $\quad p(z) = N(\mu_z, \Sigma_z)$

② obj. $\quad p(Z \mid \mu_z, \Sigma_z) = \prod_{i=1}^{N} p(z_i \mid \mu_z, \Sigma_z)$

$$= \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}^{D+1} |\Sigma_z|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(z_i - \mu_z)^T \Sigma_z^{-1} (z_i - \mu_z)\right)$$

$$\mu_z, \Sigma_z = \underset{\mu_z, \Sigma_z}{\operatorname{argmax}}\; p(Z \mid \mu_z, \Sigma_z)$$

$$\mu_z, \Sigma_z = \underset{\mu_z, \Sigma_z}{\operatorname{argmax}} \prod_{i=1}^{N} N(\mu_z, \Sigma_z)$$

$$= \underset{\mu_z, \Sigma_z}{\operatorname{argmax}} \ln \prod_{i=1}^{N} N$$

$$= \underset{\mu_z, \Sigma_z}{\operatorname{argmax}} \underbrace{\sum_{i=1}^{N} \left( -\ln \sqrt{2\pi}^{D+1} |\Sigma_z|^{\frac{1}{2}} - \frac{1}{2} (z_i - \mu_z)^T \Sigma_z^{-1} (z_i - \mu_z) \right)}_{M}$$

$$\frac{\partial M}{\partial \mu_z} = \frac{d}{d\mu_z}\left( \sum_{i=1}^{N} -\frac{1}{2} (z_i - \mu_z)^T \Sigma_z^{-1} (z_i - \mu_z) \right)$$

$$= \sum_{i=1}^{N} -\cancel{\frac{1}{2}}\, \Sigma_z^{-1} (z_i - \mu_z) = 0$$

$$\sum_{i=1}^{N} \cancel{\Sigma_z^{-1}}\, z_i = \sum_{i=1}^{N} \cancel{\Sigma_z^{-1}}\, \mu_z$$

$$= N \cdot \mu_z$$

$$\boxed{\therefore \mu_z = \frac{1}{N} \sum_{i=1}^{N} z_i}$$

$$\Sigma_z = \frac{1}{N} \sum_{i=1}^{N} (z_i - \mu_z)(z_i - \mu_z)^T$$

(PRML 2.3 93p)

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In summary

$\{x_i, y_i\}_{i=1}^{N} \xrightarrow[\text{Model}]{} N(\mu_z, \Sigma_z)$

$\xrightarrow[\text{Obj. Func.}]{} \operatorname{argmax} \sum_i \ln P(z_i \mid \mu_z, \Sigma_z)$

$\xrightarrow[\text{Learn.}]{}$ $\mu_z$ = empirical mean

$\Sigma_z$ = " cov.

$$Z = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \mu_Z = \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix} \begin{matrix} \}\, D \\ \}\, 1 \end{matrix} \qquad \Sigma_Z = \begin{array}{c|cc} & D & \\ \hline D & \Sigma_x & \Sigma_{xy} \\ & \Sigma_{yx} & \Sigma_y \end{array}$$

Inference

$$p(y \mid x) = \frac{p(x, y)}{p(x)} = N\left(\mu_y + \Sigma_{yx}\Sigma_x^{-1}(x - \mu_x),\; \Sigma_y - \Sigma_{yx}\Sigma_x^{-1}\Sigma_{xy}\right)$$

$$Z = \begin{pmatrix} X_1 \\ X_2 \\ y \end{pmatrix} \begin{array}{l} \left.\begin{array}{l} \}\, D_1 \\ \}\, D_2 \end{array}\right\} D \\ -\, 1 \end{array} \Rightarrow P(y \mid X_1)$$

$$P(X_1, X_2, y) \Rightarrow P(X_1, y) \Rightarrow P(y \mid X)$$

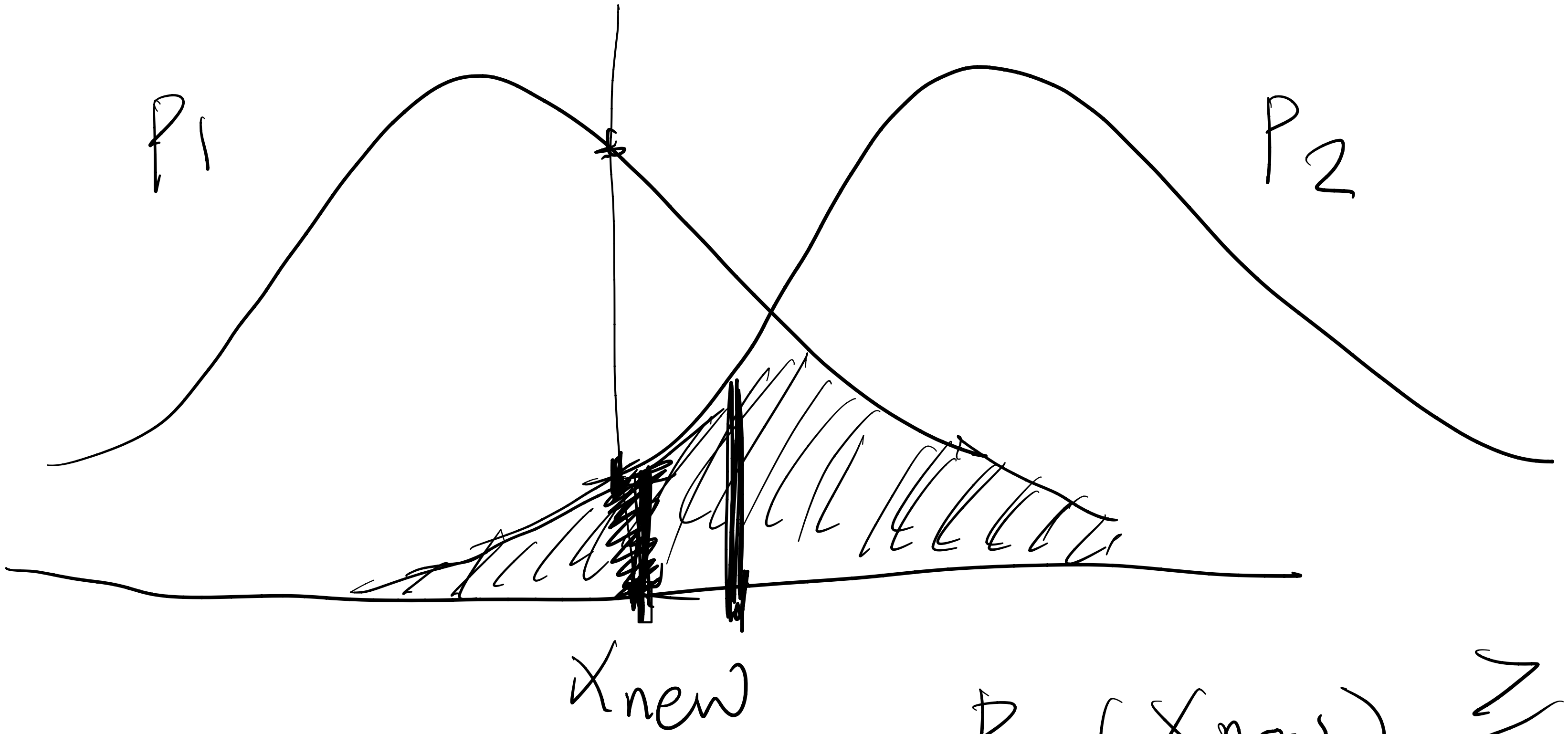
$$P(X_1, y) =$$

$$z = \begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} \qquad \mu_z = \begin{pmatrix} \mu_{x_1} \\ \mu_{x_2} \\ \mu_y \end{pmatrix} \qquad \Sigma_z = \left[\begin{array}{c|c|c} \Sigma_{x_1} & \Sigma_{x_1 x_2} & \Sigma_{x_1 y} \\ \hline \Sigma_{x_2 x_1} & \Sigma_{x_2} & \Sigma_{x_2 y} \\ \hline \Sigma_{y x_1} & \Sigma_{y x_2} & \Sigma_y \end{array}\right]$$

$$p(x_1, y) = N\left( \begin{pmatrix} \mu_{x_1} \\ \mu_y \end{pmatrix}, \begin{bmatrix} \Sigma_{x_1} & \Sigma_{x_1 y} \\ \Sigma_{y x_1} & \Sigma_y \end{bmatrix} \right)$$

$$p(y \mid x_1) = N\left( \mu_y + \Sigma_{y x_1} \Sigma_{x_1}^{-1} (x_1 - \mu_{x_1}),\ \Sigma_y - \Sigma_{y x_1} \Sigma_{x_1}^{-1} \Sigma_{x y} \right)$$

# Bayes Optimal Classifier.




$$P_1(X_{new}) \gtrless P_2(X_{new})$$

error: $$\frac{P_2(X_{new})}{P_1(X_{new}) + P_2(X_{new})}.$$

$$R^* = \int \min\left[\frac{P_1(X)}{P_1(X) + P_2(X)},\ \frac{P_2(X_2)}{P_1(X) + P_2(X)}\right] P(X)\,dX$$